OLM cl a X-a

1 Cls. X-a Problema 1

Fie $n \in \mathbb{N}^*$ dat. Calculați $[S] = ?$

unde $S = \sum_{k=1}^{2^{n+1}-1} \log_2\left(1 + \frac{1}{\sqrt{k^2+k+1}}\right)$

(Nelu Chichirim)

Soluție: $k < \sqrt{k^2+k+1} < k+1$, $\forall k \in \mathbb{N}^*$ (1p)

$\Rightarrow \log_2\left(1+\frac{1}{k}\right) > \log_2\left(1+\frac{1}{\sqrt{k^2+k+1}}\right) > \log_2\left(1+\frac{1}{k+1}\right)$ (2p)

$\Rightarrow \sum_{k=1}^{2^{n+1}-1} \log_2\left(\frac{k+1}{k}\right) > S > \sum_{k=1}^{2^{n+1}-1} \log_2\left(\frac{k+2}{k+1}\right)$ (1p)

$\Rightarrow \log_2\left(\prod_{k=1}^{2^{n+1}-1} \frac{k+1}{k}\right) > S > \log_2\left(\prod_{k=1}^{2^{n+1}-1} \frac{k+2}{k+1}\right) \Rightarrow$ (2p)

$\Rightarrow \log_2\left(2^{n+1}\right) > S > \log_2\left(\frac{2^{n+1}+1}{2}\right) \Rightarrow$

$\Rightarrow n+1 > S > \log_2\left(2^n + \frac{1}{2}\right) > \log_2\left(2^n\right) = n$ (1p)

$\Rightarrow n+1 > S > n \Rightarrow [S] = n$.

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Soluție Pb. 2

a) $f(z_1)=f(z_2) \Rightarrow \frac{z_1}{1+|z_1|}=\frac{z_2}{1+|z_2|} \Rightarrow$

$\overset{|\cdot|}{\Rightarrow} \left(\frac{|z_1|}{1+|z_1|}=\frac{|z_2|}{1+|z_2|} \Rightarrow |z_1|=|z_2| \right) \Rightarrow$ (2p)

$\Rightarrow \left\{ \begin{array}{l} 1+|z_1|=1+|z_2| \\ \frac{z_1}{1+|z_1|}=\frac{z_2}{1+|z_2|} \end{array} \right\} \Rightarrow z_1=z_2$ (2p)

b) $D=\{z\in\mathbb{C} \mid |z|<1\}$

(2p) "$\subseteq$" $|f(z)|=\frac{|z|}{|z|+1}<1$ (ev) $\Rightarrow$

$\Rightarrow f(z)\in D$

(1p) "$\supseteq$" Dacă $v\in D \Rightarrow |v|<1$.

Ec. $f(z)=v \Leftrightarrow \frac{z}{1+|z|}=v \Rightarrow$

$\Rightarrow \frac{|z|}{1+|z|}=|v| \Rightarrow |z|=\frac{|v|}{1-|v|}\geq 0$

pt că $|v|\in[0,1)$

$\frac{z}{1+|z|}=v \Leftrightarrow z=v(1+|z|) \Leftrightarrow z=v\left(1+\frac{|v|}{1-|v|}\right)$

$\Leftrightarrow \boxed{z=\frac{v}{1-|v|}}$

3 CLASA A X-A OLM

Fie $a, b \in \mathbb{R}$, $a > b > 0$;

Fie ecuația $a^x + \frac{1}{b^x} = 2$.

În ipoteza că are o rădăcină $x_0 \neq 0$, arătați că:

a) $x_0 < 0$

b) $(\exists)\ u \in \mathbb{R}$ a.î. $a^u + \frac{1}{b^u} < 2$

prof. Dorin Arventiev

Sol.

a) $2 = a^{x_0} + \frac{1}{b^{x_0}} \geq 2\sqrt{\left(\frac{a}{b}\right)^{x_0}}$ ... 2p

$\Rightarrow \left(\frac{a}{b}\right)^{x_0} \leq 1 \Rightarrow x_0 < 0$ ... 1p

b) $u = \frac{x_0}{2}$ (2p) Într-adevăr,

$a^{\frac{x_0}{2}} + b^{-\frac{x_0}{2}} < 2 \Leftrightarrow \underbrace{a^{x_0} + b^{-x_0}}_{2} + 2a^{\frac{x_0}{2}} b^{-\frac{x_0}{2}} < 4$ (1p)

$\left(\frac{a}{b}\right)^{\frac{x_0}{2}} < 1$ (A) (1p)

4

Clasa a X-a ONM

a) $z \in \mathbb{R} \Leftrightarrow z = \overline{z}$ (1p)

$\overline{z_i} = \dfrac{r^2}{z_i}$ (1p)

finalizare (2p)

b) $\left.\begin{aligned} & z_k = r(\cos t_k + i \cdot \sin t_k) \\ & z = r^2\left(1 + 8\cos\frac{t_1}{2}\cos\frac{t_2}{2}\cos\frac{t_1+t_2}{2}\right) \end{aligned}\right\}$ (2p)

finalizare (1p)